JOURNAL OF APPROXIMATION THEORY 9, 91–95 (1973)

The Müntz–Jackson Theorem in L^2

D. J. NEWMAN

Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

Communicated by Oved Shisha

Received November 15, 1971

Let Λ be a set consisting of $0 = \lambda_0$ and a finite collection of positive numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and designate by P_Λ the collection of all "polynomials" $\sum_{k=0}^{n} C_k x^{\lambda_k}$. We are concerned with quantitative measures of just how well P_Λ approximates arbitrary functions. In more precise terms we seek (best) theorems of the form

(1) To each f in $L^2[0, 1]$ there exists a P in P_Λ such that

$$\| f - P \|_2 \leqslant A\omega_2(f; \epsilon).$$

Here $\| \ \|_2$ is the L^2 norm on $[0, 1]$, ω_2 is the L^2 modulus of continuity there, and A is an absolute constant.

In short, our problem is to determine (to within absolute constant multiples) the best, i.e., smallest, ϵ, call it ϵ_Λ, which can be used in (1).

In [1] this problem is considered, and a complete solution is given when $\lambda_{k+1} - \lambda_k \geqslant 2$ for all k. This condition is a natural one when the λ_k are integers since it turns out that the even integers have the same "speed" of approximation as that of all integers and only sequences "thinner" than the even integers offer any interest. Be that as it may, the result under the above separation hypothesis is given by the very nice formula

$$\epsilon_\Lambda = e^{-2 \sum_{k=1}^{n} 1/\lambda_k}.$$

Of course, when the λ_k are not integers this separation condition is no longer natural and one wonders whether any formulas can be given for the general case. The answer is yes, and that is the purpose of this note. Unfortunately the general formula is not as explicit as that of the separated case above, but there is a sort of complementary "unseparated" case wherein this general formula again takes a pleasant explicit form.

Since ϵ_Λ is only defined within absolute constant factors we introduce the notation $\doteq$ to mean equality within absolute (positive) constant factors. In these terms our results are as follows.

THEOREM I.

$$\epsilon_\Lambda{}^2 \doteq \max_x \frac{1}{x^2 + \frac{1}{4}} \prod_\Lambda \frac{x^2 + (\lambda - \frac{1}{2})^2}{x^2 + (\lambda + \frac{3}{2})^2}.$$

THEOREM II. *If* $\lambda_{k+1} - \lambda_k \leqslant 2$ *for all* k *then* $\epsilon_\Lambda \doteq 1/(\sum_\Lambda(\lambda + \frac{1}{2}))^{1/2}$.

Special cases of Theorem II are interesting. If $\lambda_k = k^\alpha$, $0 < \alpha \leqslant 1$, then we obtain $\epsilon_\Lambda \doteq n^{-(1+\alpha)/2}$. Again if the λ_k are bounded by some fixed constant then we obtain $\epsilon_\Lambda \doteq 1/n^{1/2}$. In the limiting situation of this bounded case when the positive λ's all become equal, to $\beta > 0$ say, our theorem tells us that

$$\left\| f(x) - \left((C_0 + x^\beta \sum_{k=1}^n C_k(\log x)^k\right)\right\|_2 \leqslant A\omega_2(f, 1/n^{1/2})$$

for appropriate C_k, and that the $1/n^{1/2}$ is best possible. Setting $x = e^{-t^2}$ in this estimate leads to the quantitative form of the theorem on expansion into Hermite polynomials.

One more remark is instructive. It will be noted in our proof of Theorem II that the hypothesis on the λ's is not used in obtaining the lower bound $\epsilon_\Lambda \geqslant C/(\sum_\Lambda(\lambda + \frac{1}{2}))^{1/2}$ so that this inequality is universal. Of course, for the upper bound, our condition is very necessary. For example when $\lambda_k = (2 + \delta)k$ the correct value of ϵ_Λ, as given by [1], is $n^{-2(2+\delta)}$ while $1/(\sum (\lambda + \frac{1}{2}))^{1/2}$ is of the order n^{-1}.

Before proceeding to the proofs we find it handy to introduce some notation. First of all, the numbers $\lambda + \frac{1}{2}$ turn out to be more natural than the λ themselves, and so we set $\gamma_k = \lambda_k + \frac{1}{2}$ and we call the set of γ_k, Γ. Next we write

$$H(x) = \frac{1}{x^2 + \frac{1}{4}} \prod_\Lambda \frac{x^2 + (\lambda - \frac{1}{2})^2}{x^2 + (\lambda + \frac{3}{2})^2} = \frac{1}{x^2 + \frac{1}{4}} \prod_\Gamma \frac{x^2 + (\gamma - 1)^2}{x^2 + (\gamma + 1)^2}$$

and we call $M = \max_x H(x)$.

Proof of Theorem I. Our starting point is the formula given in [1], namely

$$\epsilon_\Lambda{}^2 = \sup_{f \in \mathscr{P}} \int_{-\infty}^{\infty} |f(x + i)|^2 H(x)\, dx, \tag{2}$$

where $\mathscr{P}$ is the Paley–Wiener class consisting of functions $f(z)$ analytic in $\operatorname{Im} z > 0$ and satisfying $\int_{-\infty}^{\infty} |f(x + iy)|^2\, dx \leqslant 1$ for all $y > 0$.

From (2) we immediately read off the upper bound $\epsilon_\Lambda^2 \leqslant M$ and so all that is needed is an $f \in \mathscr{P}$ for which $\int_{-\infty}^{\infty} |f(x+i)|^2 H(x)\,dx \geqslant cM$. Let $t \geqslant 0$ be a maximum point of $H(x)$ so that $H(t) = M$. We will choose $f(x) = \pi^{-1/2}/(x - t + i)$. This function clearly lies in $\mathscr{P}$, and we will prove that

$$\int_{-\infty}^{\infty} \frac{H(x)\,dx}{(x-t)^2 + 4} \geqslant \frac{\pi}{8e^4} M, \tag{3}$$

which gives our required lower bound with $c = 1/8e^4$.

The crucial observation is that $H'(x)/H(x) \geqslant -2$ for $x \geqslant 0$. We have, namely,

$$\begin{aligned}
\frac{H'(x)}{H(x)} &= \sum_{\Gamma} \left(\frac{2x}{x^2 + (\gamma-1)^2} - \frac{2x}{x^2 + (\gamma+1)^2} \right) - \frac{2x}{x^2 + \frac{1}{4}} \\
&= 8x \sum_{\Gamma} \frac{\gamma}{(x^2 + (\gamma-1)^2)(x^2 + (\gamma+1)^2)} - \frac{2x}{x^2 + \frac{1}{4}} \\
&\geqslant -\frac{2x}{x^2 + \frac{1}{4}} \geqslant -2.
\end{aligned}$$

Integrating this inequality from t to $t + u$ gives $\log H(t+u)/M \geqslant -2u$ or $H(t+u) \geqslant Me^{-2u}$. In particular we conclude that $H(x) \geqslant Me^{-4}$ throughout $[t, t+2]$. We, therefore, have

$$\int_{-\infty}^{\infty} \frac{H(x)\,dx}{(x-t)^2+4} \geqslant \int_{t}^{t+2} \frac{H(x)\,dx}{(x-t)^2+4} \geqslant Me^{-4} \int_{t}^{t+2} \frac{dx}{(x-t)^2+4} = \frac{\pi M}{8e^4}.$$

The proof of (3), and hence of our theorem, is complete.

Proof of Theorem II. We use the result of Theorem I which reduces the problem to showing that $C_1/S \leqslant M \leqslant C_2/S$, where $S = \sum_\Gamma \gamma$. Indeed we will prove this with $C_1 = 1/16$, $C_2 = e/4$.

To prove the first of these inequalities choose $x = (8S - \frac{1}{4})^{1/2}$ so that

$$\begin{aligned}
H(x) &= \frac{1}{8S} \prod_{\Gamma} \left(1 - \frac{4\gamma}{x^2 + (\gamma+1)^2}\right) \geqslant \frac{1}{8S} \prod_{\Gamma} \left(1 - \frac{4\gamma}{x^2 + \frac{1}{4}}\right) \\
&= \frac{1}{8S} \prod_{\Gamma} \left(1 - \frac{\gamma}{2S}\right).
\end{aligned}$$

Now we invoke the elementary fact that $\prod(1-a) \geqslant 1 - \sum a$ provided that all the a lie in $[0, 1]$. This is easily proved by induction (or by probability considerations) and applied to our case it gives

$$H(x) \geqslant \frac{1}{8S}\left(1 - \frac{\sum \gamma}{2S}\right) = \frac{1}{8S}\left(1 - \frac{S}{2S}\right) = \frac{1}{16S}$$

as promised.

As for the upper bound we select a subset, G, of Γ by choosing its first element as $\gamma = \frac{1}{2}$ and then, at each stage, choosing the largest element which is no more than two larger than the previous one chosen.

Calling these elements $\frac{1}{2} = g_0 < \cdots < g_v$, then, we observe the following facts

$$g_v = \gamma_n \text{ (the largest of the } \gamma\text{'s)}, \tag{4}$$

$$g_k - g_{k-1} \leqslant 2, \tag{5}$$

$$\sum_G \gamma \leqslant (\gamma_n + 1)^2/2. \tag{6}$$

This last follows from the facts that $g_v, g_{v-1} \leqslant \gamma_n$; $g_{v-2}, g_{v-3} \leqslant g_v - 2 \leqslant \gamma_n - 2$; $g_{v-4}, g_{v-5} \leqslant g_{v-2} - 2 \leqslant \gamma_n - 4$, etc. so that

$$\sum_G \gamma \leqslant 2\gamma_n + 2(\gamma_n - 2) + \cdots \leqslant (\gamma_n + 1)^2/2.$$

Now we have, by (4) and (5),

$$\prod_G \frac{x^2 + (\gamma - 1)^2}{x^2 + (\gamma + 1)^2} = \frac{x^2 + \frac{1}{4}}{x^2 + (\gamma_n + 1)^2} \prod_{k=1}^{v} \frac{x^2 + (g_k - 1)^2}{x^2 + (g_{k-1} + 1)^2}$$
$$\leqslant \frac{x^2 + \frac{1}{4}}{x^2 + (\gamma_n + 1)^2}. \tag{7}$$

While, on the other hand, using the elementary fact that $1 - u \leqslant e^{-u}$, we have

$$\prod_{\Gamma - G} \frac{x^2 + (\gamma - 1)^2}{x^2 + (\gamma + 1)^2} = \prod_{\Gamma - G} \left(1 - \frac{4\gamma}{x^2 + (\gamma + 1)^2}\right)$$
$$\leqslant \prod_{\Gamma - G} \left(1 - \frac{4\gamma}{x^2 + (\gamma_n + 1)^2}\right)$$
$$\leqslant \exp\left(\frac{-4}{x^2 + (\gamma_n + 1)^2} \sum_{\Gamma - G} \gamma\right)$$
$$= \exp\left(\frac{4S}{x^2 + (\gamma_n + 1)^2}\right) \exp\left(\frac{4}{x^2 + (\gamma_n + 1)^2} \sum_G \gamma\right)$$
$$\leqslant \exp\left(\frac{4S}{x^2 + (\gamma_n + 1)^2}\right) \cdot e^2 \qquad \text{by (6)}. \tag{8}$$

Finally, multiplying (7) by (8) gives

$$\frac{1}{x^2 + \frac{1}{4}} \prod_\Gamma \frac{x^2 + (\gamma - 1)^2}{x^2 + (\gamma + 1)^2} \leqslant \frac{e^2}{4S} \cdot \frac{4S}{x^2 + (\gamma_n + 1)^2} \exp\left(\frac{-4S}{x^2 + (\gamma_n + 1)^2}\right),$$

and the proof is completed by the elementary fact that te^{-t} has e^{-1} as its maximum. Q.E.D.

Reference

1. D. J. Newman, A Müntz–Jackson theorem, *Amer. J. Math.* **88** (1965), 940–944.

Printed in Belgium